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Maxwell's Treatise, part IV, Ch. 9

Premessa

Quella che segue è la trascrizione del Cap. 9 della parte II del *Treatise* di Maxwell, presa dalla scansione della prima edizione (1873) disponibile in rete.

Ho conservato con la massima fedeltà che mi è stata possibile la stesura originale, anche negli aspetti tipografici. In particolare ho lasciato le derivate come le ha scritte Maxwell, anche dove avrebbe dovuto usare i simboli di derivata parziale (che ai tempi di Maxwell erano già in uso da tempo). Gli unici cambiamenti che ho apportato sono:

- ho sostituito le lettere gotiche maiuscole che Maxwell usò per indicare vettori con la corrispondente maiuscola nel tipo "calligrafico" di TeX
- ho aggiunto una numerazione (fra parentesi quadra) alle equazioni nel riepilogo finale, per poterle richiamare nel commento che si trova appresso.

Il commento aveva inizialmente il solo scopo di facilitare la lettura, mostrando la scrittura moderna della stesse equazioni e ricordando a chi non l'avesse immediatamente presente il posto che ciascuna equazione occupa nella teoria.

Tuttavia strada facendo le cose si sono complicate, perché l'impostazione di Maxwell in diversi punti differisce parecchio da quella oggi tradizionale: mi sono quindi trovato costretto a un'interpretazione e un confronto. Ho cercato di essere più stringato possibile, anche se forse ho dovuto sacrificare un po' la comprensibilità. Purtroppo non era possibile fare diversamente, a meno di non cambiare completamente prospettiva, scrivendo un articolo storico-critico. Cosa che non era e non è nelle mie intenzioni.

Dopo il commento ho inserito una "Appendice sui quaternioni": sebbene non indispensabile per comprendere il testo, può aiutare chi non abbia familiarità con questa struttura, oggi poco usata, e voglia vedere meglio la relazione tra la forma data da Maxwell alle sue equazioni e quella poi affermatasi fino ai nostri giorni.

CHAPTER IX.

GENERAL EQUATIONS OF THE ELECTROMAGNETIC FIELD.

604.] In our theoretical discussion of electrodynamics we began by assuming that a system of circuits carrying electric currents is a dynamical system, in which the currents may be regarded as velocities, and in which the coordinates corresponding to these velocities do not themselves appear in the equations. It follows from this that the kinetic energy of the system, so far as it depends on the currents, is a homogeneous quadratic function of the currents, in which the coefficients depend only on the form and relative position of the circuits. Assuming these coefficients to be known, by experiment or otherwise, we deduced, by purely dynamical reasoning, the laws of the induction of currents, and of electromagnetic attraction. In this investigation we introduced the conceptions of the electrokinetic energy of a system of currents, of the electromagnetic momentum of a circuit, and of the mutual potential of two circuits.

We then proceeded to explore the field by means of various configurations of the secondary circuit, and were thus led to the conception of a vector $\mathcal{A}$, having a determinate magnitude and direction at any given point of the field. We called this vector the electromagnetic momentum at that point. This quantity may be considered as the time-integral of the electromotive force which would be produced at that point by the sudden removal of all the currents from the field. It is identical with the quantity already investigated in Art. 405 as the vector-potential of magnetic induction. Its components parallel to x, y, and z are F, G, and H. The electromagnetic momentum of a circuit is the line-integral of $\mathcal{A}$ round the circuit.

We then, by means of Theorem IV, Art. 24, transformed the line-integral of $\mathcal{A}$ into the surface-integral of another vector, $\mathcal{B}$, whose components are a, b, c, and we found that the phenomena of induction due to motion of a conductor, and those of electromagnetic force can be expressed in terms of $\mathcal{B}$. We gave to $\mathcal{B}$ the name of the Magnetic induction, since its properties are identical with those of the lines of magnetic induction as investigated by Faraday.

We also established three sets of equations: the first set, (A), are those of magnetic induction, expressing it in terms of the electromagnetic momentum. The second set, (B), are those of electromotive force, expressing it in terms of the motion of the conductor across the lines of magnetic induction, and of the rate of variation of the electromagnetic momentum. The third set, (C), are the equations of electromagnetic force, expressing it in terms of the current and the magnetic induction.

The current in all these cases is to be understood as the actual current, which includes not only the current of conduction, but the current due to variation of the electric displacement.

The magnetic induction $\mathcal{B}$ is the quantity which we have already considered in Art. 400. In an unmagnetized body it is identical with the force on a unit magnetic pole, but if the body is magnetized, either permanently or by induction, it is the force which would be exerted on a unit pole, if placed in a narrow crevasse in the body, the walls of which are perpendicular to the direction of magnetization. The components of $\mathcal{B}$ are a, b, c.

It follows from the equations (A), by which a, b, c are defined, that

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0.$$

This was shewn at Art. 403 to be a property of the magnetic induction.

605.] We have defined the magnetic force within a magnet, as distinguished from the magnetic induction, to be the force on a unit pole placed in a narrow crevasse cut parallel to the direction of magnetization. This quantity is denoted by $\mathcal{H}$, and its components by α, β, γ. See Art. 398.

If $\mathcal{I}$ is the intensity of magnetization, and A, B, C its components, then, by Art. 400,

$$\left.\begin{aligned} a &= \alpha + 4\pi A, \\ b &= \beta + 4\pi B, \\ c &= \gamma + 4\pi C. \end{aligned}\right\} \quad \text{(Equations of Magnetization.)} \qquad \text{(D)}$$

We may call these the equations of magnetization, and they indicate that in the electromagnetic system the magnetic induction $\mathcal{B}$, considered as a vector, is the sum, in the Hamiltonian sense, of two vectors, the magnetic force $\mathcal{H}$, and the magnetization $\mathcal{I}$ multiplied by 4π, or $\mathcal{B} = \mathcal{H} + 4\pi\mathcal{I}$. In certain substances, the magnetization depends on the magnetic force, and this is expressed by the system of equations of induced magnetism given at Arts. 426 and 435.

606.] Up to this point of our investigation we have deduced everything from purely dynamical considerations, without any reference to quantitative experiments in electricity or magnetism. The only use we have made of experimental knowledge is to recognise, in the abstract quantities deduced from the theory, the concrete quantities discovered by experiment, and to denote them by names which indicate their physical relations rather than their mathematical generation.

In this way we have pointed out the existence of the electromagnetic momentum $\mathcal{A}$ as a vector whose direction and magnitude vary from one part of space to another, and from this we have deduced, by a mathematical process, the magnetic induction, $\mathcal{B}$, as a derived vector. We have not, however, obtained any data for determining either $\mathcal{A}$ or $\mathcal{B}$ from the distribution of currents in the field. For this purpose we must find the mathematical connexion between these quantities and the currents.

We begin by admitting the existence of permanent magnets, the mutual action of which satisfies the principle of the conservation of energy. We make no assumption with respect to the laws of magnetic force except that which follows from this principle, namely, that the force acting on a magnetic pole must be capable of being derived from a potential.

We then observe the action between currents and magnets, and we find that a current acts on a magnet in a manner apparently the same as another magnet would act if its strength, form, and position were properly adjusted, and that the magnet acts on the current in the same way as another current. These observations need not be supposed to be accompanied with actual measurements of the forces. They are not therefore to be considered as furnishing numerical data, but are useful only in suggesting questions for our consideration.

The question these observations suggest is, whether the magnetic field produced by electric currents, as it is similar to that produced by permanent magnets in many respects, resembles it also in being related to a potential?

The evidence that an electric circuit produces, in the space surrounding it, magnetic effects precisely the same as those produced by a magnetic shell bounded by the circuit, has been stated in Arts. 482-485.

We know that in the case of the magnetic shell there is a potential, which has a determinate value for all points outside the substance of the shell, but that the values of the potential at two neighbouring points, on opposite sides of the shell, differ by a finite quantity.

If the magnetic field in the neighbourhood of an electric current resembles that in the neighbourhood of a magnetic shell, the magnetic potential, as found by a line-integration of the magnetic force, will be the same for any two lines of integration, provided one of these lines can be transformed into the other by continuous motion without cutting the electric current.

If, however, one line of integration cannot be transformed into the other without cutting the current, the line-integral of the magnetic force along the one line will differ from that along the other by a quantity depending on the strength of the current. The magnetic potential due to an electric current is therefore a function having an infinite series of values with a common difference, the particular value depending on the course of the line of integration. Within the substance of the conductor, there is no such thing as a magnetic potential.

607.] Assuming that the magnetic action of a current has a magnetic potential of this kind, we proceed to express this result mathematically.

In the first place, the line-integral of the magnetic force round any closed curve is zero, provided the closed curve does not surround the electric current.

In the next place, if the current passes once, and only once, through the closed curve in the positive direction, the line integral has a determinate value, which may be used as a measure of the strength of the current. For if the closed curve alters its form in any continuous manner without cutting the current, the line-integral will remain the same.

In electromagnetic measure, the line-integral of the magnetic force round a closed curve is numerically equal to the current through the closed curve multiplied by 4π.

If we take for the closed curve the parallelogram whose sides are dy and dz, the line-integral of the magnetic force round the parallelogram is

$$\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) dy\, dz,$$

and if u, v, w are the components of the flow of electricity, the current through the parallelogram is

$$u\, dy\, dz.$$

Multiplying this by 4π, and equating the result to the line-integral, we obtain the equation

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

with the similar equations

$$\left.\begin{aligned} 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy}, \end{aligned}\right\} \text{(Equations of Electric Currents.)} \qquad \text{(E)}$$

which determine the magnitude and direction of the electric currents when the magnetic force at every point is given.

When there is no current, these equations are equivalent to the condition that

$$\alpha\, dx + \beta\, dy + \gamma\, dz = -\mathrm{D}\,\Omega,$$

or that the magnetic force is derivable from a magnetic potential in all points of the field where there are no currents.

By differentiating the equations (E) with respect to x, y, and z respectively, and adding the results, we obtain the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

which indicates that the current whose components are u, v, w is subject to the condition of motion of an incompressible fluid, and that it must necessarily flow in closed circuits.

This equation is true only if we take u, v, and w as the components of that electric flow which is due to the variation of electric displacement as well as to true conduction.

We have very little experimental evidence relating to the direct electromagnetic action of currents due to the variation of electric displacement in dielectrics, but the extreme difficulty of reconciling the laws of electromagnetism with the existence of electric currents which are not closed is one reason among many why we must admit the existence of transient currents due to the variation of displacement. Their importance will be seen when we come to the electromagnetic theory of light.

608.] We have now determined the relations of the principal quantities concerned in the phenomena discovered by Örsted, Ampère, and Faraday. To connect these with the phenomena described in the former parts of this treatise, some additional relations are necessary.

When electromotive force acts on a material body, it produces in it two electrical effects, called by Faraday Induction and Conduction, the first being most conspicuous in dielectrics, and the second in conductors.

In this treatise, static electric induction is measured by what we have called the electric displacement, a directed quantity or vector which we have denoted by $\mathcal{D}$, and its components by f, g, h.

In isotropic substances, the displacement is in the same direction as the electromotive force which produces it, and is proportional to it, at least for small values of this force. This may be expressed by the equation

$$\mathcal{D} = \frac{1}{4\pi} K \mathcal{E}, \qquad \text{(Equation of Electric Displacement.)} \tag{F}$$

where K is the dielectric capacity of the substance. See Art. 69.

In substances which are not isotropic, the components f, g, h of the electric displacement $\mathcal{D}$ are linear functions of the components P, Q, R of the electromotive force $\mathcal{E}$.

The form of the equations of electric displacement is similar to that of the equations of conduction as given in Art. 298.

These relations may be expressed by saying that K is, in isotropic bodies, a scalar quantity, but in other bodies it is a linear and vector function, operating on the vector $\mathcal{E}$.

609.] The other effect of electromotive force is conduction. The laws of conduction as the result of electromotive force were established by Ohm, and are explained in the second part of this treatise, Art. 241. They may be summed up in the equation

$$\mathcal{K} = C \mathcal{E}, \qquad \text{(Equation of Conductivity.)} \tag{G}$$

where $\mathcal{E}$ is the intensity of the electromotive force at the point, $\mathcal{K}$ is the density of the current of conduction, the components of which are p, q, r, and C is the conductivity of the substance, which, in the case of isotropic substances, is a simple scalar quantity, but in other substances becomes a linear and vector function operating on the vector $\mathcal{E}$. The form of this function is given in Cartesian coordinates in Art. 298.

610.] One of the chief peculiarities of this treatise is the doctrine which it asserts, that the true electric current $\mathcal{C}$, that on which the electromagnetic phenomena depend, is not the same thing as $\mathcal{K}$, the current of conduction, but that the time-variation of $\mathcal{D}$, the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write,

$$\mathcal{C} = \mathcal{K} + \dot{\mathcal{D}} \quad \text{(Equation of True Currents.)} \tag{H}$$

or, in terms of the components,

$$\left.\begin{aligned} u &= p + \frac{df}{dt}, \\ v &= q + \frac{dg}{dt}, \\ w &= r + \frac{dh}{dt}. \end{aligned}\right\} \tag{H$*$}$$

611.] Since both $\mathcal{K}$ and $\mathcal{D}$ depend on the electromotive force $\mathcal{E}$, we may express the true current $\mathcal{C}$ in terms of the electromotive force, thus

$$\mathcal{C} = \left(C + \frac{1}{4\pi} K \frac{d}{dt}\right)\mathcal{E}, \tag{I}$$

or, in the case in which C and K are constants,

$$\left.\begin{aligned} u &= C\,P + \frac{1}{4\pi} K \frac{dP}{dt}, \\ v &= C\,Q + \frac{1}{4\pi} K \frac{dQ}{dt}, \\ w &= C\,R + \frac{1}{4\pi} K \frac{dR}{dt}. \end{aligned}\right\} \tag{I$*$}$$

612.] The volume-density of the free electricity at any point is found from the components of electric displacement by the equation

$$\varrho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}. \tag{J}$$

613.] The surface-density of electricity is

$$\sigma = lf + mg + nh + l'f' + m'g' + n'h', \tag{K}$$

where l, m, n are the direction-cosines of the normal drawn from the surface into the medium in which f, g, h are the components of the displacement, and l', m', n' are those of the normal drawn from the surface into the medium in which they are f', g', h'.

614.] When the magnetization of the medium is entirely induced by the magnetic force acting on it, we may write the equation of induced magnetization,

$$\mathcal{B} = \mu \mathcal{H}, \tag{L}$$

where μ is the coefficient of magnetic permeability, which may be considered a scalar quantity, or a linear and vector function operating on $\mathcal{H}$, according as the medium is isotropic or not.

615.] These may be regarded as the principal relations among the quantities we have been considering. They may be combined so as to eliminate some of these quantities, but our object at present is not to obtain compactness in the mathematical formulae, but to express every relation of which we have any knowledge. To eliminate a quantity which expresses a useful idea would be rather a loss than a gain in this stage of our enquiry.

There is one result, however, which we may obtain by combining equations (A) and (E), and which is of very great importance.

If we suppose that no magnets exist in the field except in the form of electric circuits, the distinction which we have hitherto maintained between the magnetic force and the magnetic induction vanishes, because it is only in magnetized matter that these quantities differ from each other.

According to Ampère's hypothesis, which will be explained in Art. 833, the properties of what we call magnetized matter are due to molecular electric circuits, so that it is only when we regard the substance in large masses that our theory of magnetization is applicable, and if our mathematical methods are supposed capable of taking account of what goes on within the individual molecules, they will discover nothing but electric circuits, and we shall find the magnetic force and the magnetic induction everywhere identical. In order, however, to be able to make use of the electrostatic or of the electromagnetic system of measurement at pleasure we shall retain the coefficient μ, remembering that its value is unity in the electromagnetic system.

616.] The components of the magnetic induction are by equations (A), Art. 591,

$$\left.\begin{aligned} a &= \frac{dH}{dy} - \frac{dG}{dz}, \\ b &= \frac{dF}{dz} - \frac{dH}{dx}, \\ c &= \frac{dG}{dx} - \frac{dH}{dy}. \end{aligned}\right\}$$

The components of the electric current are by equations (E), Art. 607,

$$\left.\begin{aligned} 4\pi u &= \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \\ 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy}. \end{aligned}\right\}$$

According to our hypothesis a, b, c are identical with $\mu\,\alpha$, $\mu\,\beta$, $\mu\,\gamma$ respectively. We therefore obtain

$$4\pi\mu\,u = \frac{d^2G}{dx\,dy} - \frac{d^2F}{dy^2} - \frac{d^2F}{dz^2} + \frac{d^2H}{dz\,dx}. \tag{1}$$

If we write

$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}, \tag{2}$$

and*

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right). \tag{3}$$

we may write equation (1),

Similarly,

$$\left.\begin{aligned} 4\pi\,\mu\,u &= \frac{dJ}{dx} + \nabla^2 F. \\ 4\pi\,\mu\,v &= \frac{dJ}{dy} + \nabla^2 G, \\ 4\pi\,\mu\,w &= \frac{dJ}{dz} + \nabla^2 H. \end{aligned}\right\} \tag{4}$$

*The negative sign is employed here in order to make our expressions consistent with those in which Quaternions are employed.

If we write

$$\left.\begin{aligned} F' &= \frac{1}{\mu}\iiint \frac{u}{r}\,dx\,dy\,dz, \\ G' &= \frac{1}{\mu}\iiint \frac{v}{r}\,dx\,dy\,dz, \\ H' &= \frac{1}{\mu}\iiint \frac{w}{r}\,dx\,dy\,dz, \end{aligned}\right\} \tag{5}$$

$$\chi = \frac{4\pi}{\mu}\iiint \frac{J}{r}\,dx\,dy\,dz, \tag{6}$$

where r is the distance of the given point from the element x, y, z, and the integrations are to be extended over all space, then

$$\left.\begin{aligned} F &= F' + \frac{d\chi}{dx}, \\ G &= G' + \frac{d\chi}{dy}, \\ H &= H' + \frac{d\chi}{dz}. \end{aligned}\right\} \tag{7}$$

The quantity χ disappears from the equations (A), and it is not related to any physical phenomenon. If we suppose it to be zero everywhere, J will also be zero everywhere, and equations (5), omitting the accents, will give the true values of the components of $\mathcal{A}$.

617.] We may therefore adopt, as a definition of $\mathcal{A}$, that it is the vector-potential of the electric current, standing in the same relation to the electric current that the scalar potential stands to the matter of which it is the potential, and obtained by a similar process of integration, which may be thus descrihed.

From a given point let a vector be drawn, representing in magnitude and direction a given element of an electric current, divided by the numerical value of the distance of the element from the given point. Let this be done for every element of the electric current. The resultant of all the vectors thus found is the potential of the whole current. Since the current is a vector quantity, its potential is also a vector. See Art. 422.

When the distribution of electric currents is given, there is one, and only one, distribution of the values of $\mathcal{A}$, such that $\mathcal{A}$ is everywhere finite and continuous, and satisfies the equations

$$\nabla^2\mathcal{A} = 4\pi\,\mu\,\mathcal{C}, \qquad \mathrm{S}\,\nabla\mathcal{A} = 0,$$

and vanishes at an infinite distance from the electric system. This value is that given by equations (5), which may be written

$$\mathcal{A} = \frac{1}{\mu}\iiint \frac{\mathcal{C}}{r}\,dx\,dy\,dz.$$

Quaternion Expressions for the Electromagnetic Equations.

618.] In this treatise we have endeavoured to avoid any process demanding from the reader a knowledge of the Calculus of Quaternions. At the same time we have not scrupled to introduce the idea of a vector when it was necessary to do so. When we have had occasion to denote a vector by a symbol, we have used a German letter, the number of different vectors being so great that Hamilton's favourite symbols would have been exhausted at once. Whenever therefore, a German letter is used it denotes a Hamiltonian vector, and indicates not only its magnitude but its direction. The constituents of a vector are denoted by Roman or Greek letters.

The principal vectors which we have to consider are :—

	Symbol of Vector.	Constituents.		
The radius vector of a point	ϱ	x	y	z
The electromagnetic momentum of a point	$\mathcal{A}$	F	G	H
The magnetic induction	$\mathcal{B}$	a	b	c
The (total) electric current	$\mathcal{C}$	u	v	w
The electric displacement	$\mathcal{D}$	f	g	h
The electromotive force	$\mathcal{E}$	P	Q	R
The mechanical force	$\mathcal{F}$	X	Y	Z
The velocity of a point	$\mathcal{G}$ or $\dot{\varrho}$	$\dot{x}$	$\dot{y}$	$\dot{z}$
The magnetic force	$\mathcal{H}$	α	β	γ
The intensity of magnetization	$\mathcal{I}$	A	B	C
The current of conduction	$\mathcal{K}$	p	q	r

We have also the following scalar functions :–

The electric potential Ψ.

The magnetic potential (where it exists) Ω.

The electric density e.

The density of magnetic 'matter' m.

Besides these we have the following quantities, indicating physical properties of the medium at each point :—

C, the conductivity for electric currents.

K, the dielectric inductive capacity.

μ, the magnetic inductive capacity.

These quantities are, in isotropic media, mere scalar functions of ϱ, but in general they are linear and vector operators on the vector functions to which they are applied. K and μ are certainly always self-conjugate, and C is probably so also.

619.] The equations (A) of magnetic induction, of which the first is,

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

may now be written

$$\mathcal{B} = \mathrm{V}\,\nabla\mathcal{A}, \qquad [1]$$

where ∇ is the operator

$$i\,\frac{d}{dx} + j\,\frac{d}{dy} + k\,\frac{d}{dz},$$

and V indicates that the vector part of the result of this operation is to be taken.

Since $\mathcal{A}$ is subject to the condition $\mathrm{S}\,\nabla\mathcal{A} = 0$, $\nabla\mathcal{A}$ is a pure vector, and the symbol V is unnecessary.

The equations (B) of electromotive force, of which the first is

$$P = c\,\dot{y} - b\,\dot{z} - \frac{dF}{dt} - \frac{d\Psi}{dx},$$

become

$$\mathcal{E} = \mathrm{V}\,\mathcal{G}\,\mathcal{B} - \dot{\mathcal{A}} - \nabla\Psi. \qquad [2]$$

The equations (C) of mechanical force, of which the first is

$$X = c\,v - b\,w - e\,\frac{d\Psi}{dx} - m\,\frac{d\Omega}{dx},$$

become

$$\mathcal{F} = \mathrm{V}\,\mathcal{C}\,\mathcal{B} - e\,\nabla\Psi - m\,\nabla\Omega. \qquad [3]$$

The equations (D) of magnetization, of which the first is

$$a = \alpha + 4\pi\,A,$$

become

$$\mathcal{B} = \mathcal{H} + 4\pi\,\mathcal{I}. \qquad [4]$$

The equations (E) of electric currents, of which the first is

$$4\pi\,u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

become

$$4\pi\,\mathcal{C} = \mathrm{V}\,\nabla\mathcal{H}. \qquad [5]$$

The equation of the current of conduction is, by Ohm's Law,

$$\mathcal{K} = C\,\mathcal{E}. \qquad [6]$$

That of electric displacement is

$$\mathcal{D} = \frac{1}{4\pi} K \mathcal{E}. \qquad [7]$$

The equation of the total current, arising from the variation of the electric displacement as well as from conduction, is

$$\mathcal{C} = \mathcal{K} + \dot{\mathcal{D}}. \qquad [8]$$

When the magnetization arises from magnetic induction,

$$\mathcal{B} = \mu \mathcal{H}. \qquad [9]$$

We have also, to determine the electric volume-density,

$$e = \mathrm{S}\, \nabla \mathcal{D}. \qquad [10]$$

To determine the magnetic volume-density,

$$m = \mathrm{S}\, \nabla \mathcal{I}. \qquad [11]$$

When the magnetic force can be derived from a potential

$$\mathcal{H} = -\nabla \Omega. \qquad [12]$$

Commento

Qui di seguito elenco le equazioni che si trovano nella pagina finale del Cap. 9, indicandone il significato moderno. Comincio con due delle "eq. di Maxwell" propriamente dette:

$$e = \mathrm{S}\,\nabla\mathcal{D} \qquad\qquad \nabla\cdot\mathbf{D} = \varrho \qquad [10]$$

$$4\pi\,\mathcal{C} = \mathrm{V}\,\nabla\mathcal{H} \qquad\qquad \nabla\times\mathbf{H} = 4\pi\left(\mathbf{j} + \frac{\partial\mathbf{D}}{\partial t}\right). \qquad [5]$$

Mancano, come si vede, due equazioni: quella per la divergenza di $\mathbf{B}$, e quella per il rotore di $\mathbf{E}$. La prima in realtà si trova scritta, in componenti, come eq. (18) all'Art. 403:

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0.$$

Mi sento perciò autorizzato ad aggiungerla:

$$\mathrm{S}\,\nabla\mathcal{B} = 0 \qquad\qquad \nabla\cdot\mathbf{B} = 0. \qquad [13]$$

Si noti che la [13] è conseguenza della [1]:

$$\mathcal{B} = \mathrm{V}\,\nabla\mathcal{A} \qquad\qquad \mathbf{B} = \nabla\times\mathbf{A}. \qquad [1]$$

Quanto al campo elettrico, sembrerebbe definito dalla [2] (1):

$$\mathcal{E} = \mathrm{V}\,\mathcal{G}\,\mathcal{B} - \dot{\mathcal{A}} - \nabla\Psi \qquad\qquad \mathbf{E} = \mathbf{v}\times\mathbf{B} - \frac{\partial\mathbf{A}}{\partial t} - \nabla\phi. \qquad [2]$$

Ma occorre osservare che quello che M. indica con $\mathcal{E}$ *non è* ciò che oggi viene inteso con questo nome. Basta seguire il ragionamento dell'Art. 599 per capire che M. indica con $\mathcal{E}$ il campo la cui circuitazione, calcolata a un istante fissato su un circuito in moto, fornisce la f.e.m. indotta. Nel detto articolo $\mathcal{E}$ viene diviso in tre parti:

- $\mathcal{E}_1$ dipende dal moto dell'elemento di circuito: la sua espressione (con simboli moderni) è $\mathbf{v}\times\mathbf{B}$ (forza di Lorentz).
- $\mathcal{E}_2$ dipende dalla variazione del campo magnetico: la sua espressione è $-\partial\mathbf{A}/\partial t$.
- $\mathcal{E}_3$ dipende dalla variazione spaziale del potenziale scalare ϕ, e vale $-\nabla\phi$.

(1) M. usa sistematicamente il punto, come in $\dot{\mathcal{A}}$ o in $\dot{\mathcal{D}}$, dove noi oggi scriveremmo una derivata parziale. Anche le derivate rispetto alle coordinate, che andrebbero scritte come parziali, le scrive come derivate ordinarie: si veda ad es. la divergenza di $\mathcal{B}$ scritta sopra.

Ne segue che il nostro campo elettrico include il secondo e il terzo termine, ma non il primo. Se lo chiamo $\mathcal{E}'$, dalla [2] prendendo il rotore e usando la [1] ottengo

$$\mathrm{V}\,\nabla\mathcal{E}' = -\dot{\mathcal{B}} \qquad\qquad \nabla\times\mathbf{E}' = -\frac{\partial\mathbf{B}}{\partial t} \qquad [14]$$

che è la quarta eq. di Maxwell (legge dell'induzione).

Passiamo a questioni più semplici. La [4]

$$\mathcal{B} = \mathcal{H} + 4\pi\,\mathcal{I} \qquad\qquad \mathbf{B} = \mathbf{H} + 4\pi\,\mathbf{M} \qquad [4]$$

è in uso ancor oggi con la stessa forma, ed esprime la relazione tra **B**, **H** e l'intensità di magnetizzazione **M**. Le due eq. [7] e [9]

$$\mathcal{D} = \frac{1}{4\pi}\,K\,\mathcal{E} \qquad\qquad \mathbf{D} = \frac{\varepsilon}{4\pi}\,\mathbf{E} \qquad [7]$$

$$\mathcal{B} = \mu\,\mathcal{H} \qquad\qquad \mathbf{B} = \mu\,\mathbf{H} \qquad [9]$$

note come "eq. di collegamento" o "eq. di struttura" non hanno validità generale, ma valgono solo nei casi lineari: quando la polarizzazione elettrica o risp. la magnetizzazione dipendono linearmente dai campi. Inoltre è assunta l'isotropia: questo M. lo dice esplicitamente e ricorda che altrimenti ε e μ diventano *tensori.* Naturalmente M. non usa questo termine, che avrebbe preso il significato attuale solo anni dopo [?].

Un'equazione con carattere simile alle precedenti è la [6]:

$$\mathcal{K} = C\,\mathcal{E} \qquad\qquad \mathbf{j} = \sigma\,\mathbf{E} \qquad [6]$$

che esprime in forma differenziale la legge di Ohm: la densità di corrente (di conduzione) è proporzionale al campo elettrico. Si noti che compare **E**, non $\mathbf{E}'$, ed è giusto, nel senso che se il mezzo conduttore è in moto, a muovere le cariche contribuisce anche la forza di Lorentz.

Non ho tradotto la [8]

$$\mathcal{C} = \mathcal{K} + \dot{\mathcal{D}} \qquad [8]$$

in forma moderna, perché un equivalente della "corrente totale" $\mathcal{C}$ oggi non è in uso. Infatti nella [5], dove M. usa $\mathcal{C}$, vi ho sostituito $\mathbf{j} + \partial\mathbf{D}/\partial t$, servendomi proprio della definizione [8] di $\mathcal{C}$.

Vediamo ora altre due equazioni. La prima è

$$m = \mathrm{S}\,\nabla\mathcal{I} \qquad\qquad \nabla\cdot\mathbf{M} = \varrho_{\mathrm{m}} \qquad [11]$$

dove ϱ_{m} è la "densità di carica di magnetizzazione." Sebbene le cariche magnetiche non esistano, M. introduce questo termine in analogia a quello di "densità di carica di polarizzazione elettrica." La cosa strana è che questo concetto, e anche quello di polarizzazione elettrica $\mathbf{P}$, analogo alla magnetizzazione $\mathbf{M}$, non mi pare che siano presenti nel *Treatise*. Anticipo che ϱ_{m} conparirà nella [3] che vedremo più oltre.

La seconda equazione è

$$\mathcal{H} = -\nabla\Omega \qquad\qquad \mathbf{H} = -\nabla\Omega \qquad [12]$$

Questa introduce il concetto di "potenziale magnetico" Ω, che ha senso nei casi in cui, annullandosi il secondo membro della [5], il campo magnetico è conservativo.

Ho lasciato per ultima la [3], che è un caso spinoso:

$$\mathcal{F} = \mathrm{V}\,\mathcal{C}\,\mathcal{B} - e\,\nabla\Psi - m\,\nabla\Omega \qquad \mathbf{f} = \mathbf{j}\times\mathbf{B} - \varrho\,\nabla\phi - \varrho_{\mathrm{m}}\nabla\Omega. \qquad [3]$$

Questa è chiamata da M. "equazione della forza meccanica." Faccio anzitutto notare come ho trascritto le notazioni. Dove M. usa $\mathcal{C}$, ho sostituito $\mathbf{j}$, che non è la stessa cosa; spiegazione tra poco. Dove scrive $\mathcal{F}$ ho preferito la minuscola $\mathbf{f}$: questo perché si tratta della *densità di forza* (forza per unità di volume) agente su un sistema continuo che possiede una densità di carica elettrica ϱ, una densità di carica magnetica ϱ_{m} e porta una densità di corrente elettrica $\mathbf{j}$.

Secondo M. la (densità di) forza in presenza di campo magnetico è dovuta non a $\mathbf{j}$ ma a $\mathbf{j} + \partial\mathbf{D}/\partial t$. La giustificazione si trova all'Art. 604, dove si dice che "la corrente in tutti questi casi va intesa come la corrente effettiva, che include non solo la corrente di conduzione, ma anche la corrente dovuta alla variazione dell'induzione elettrica ($\mathbf{D}$)." Ed è tutto, quanto a spiegazione.

I casi di cui si tratta sono appunto la forza elettromotrice [5] e la forza meccanica [3]. Della [5] ho già detto; quanto alla [3], non vedo come si possa motivare l'intervento di $\partial\mathbf{D}/\partial t$. Si dovrebbe supporre che per es. un dielettrico polarizzabile in presenza di un campo elettrico variabile (che genera una $\partial\mathbf{D}/\partial t$ non nulla) e di un campo magnetico $\mathbf{B}$ risenta una forza. A me questo non sembra vero, o per meglio dire lo accetterei se al posto di $\partial\mathbf{D}/\partial t$ ci fosse $\partial\mathbf{P}/\partial t$.

Ma lasciamo la questione in sospeso, almeno per ora.

Nell'Art. 607 è anche dedotta l'equazione di continuità, scritta

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Dato che u, v, w sono le componenti della corrente totale $\mathcal{C}$, questa si traduce

$$\mathrm{S}\,\nabla\mathcal{K} + \mathrm{S}\,\nabla\dot{\mathcal{D}} = 0 \qquad\qquad \nabla\cdot\mathbf{j} + \nabla\cdot\frac{\partial\mathbf{D}}{\partial t} = 0$$

e usando la [10]:

$$\mathrm{S}\,\nabla\mathcal{K} + \dot{e} = 0 \qquad\qquad \nabla\cdot\mathbf{j} + \frac{\partial\varrho}{\partial t} = 0. \qquad [13]$$

Eq. che non compare nel *Treatise*, ma è conseguenza immediata di eq. che sono presenti. Per questo motivo l'ho aggiunta alla lista.

Appendice sui quaternioni

Si possono seguire diversi approcci per introdurre i quaternioni. Qui, dato lo scopo di questo scritto, la via migliore è forse quella più vicina all'originaria di Hamilton, ossia come generalizzazione del campo complesso.

Chiamerò dunque *quaternione* una scrittura del tipo

$$Q = a + b\,i + c\,j + d\,k$$

dove a, b, c, d sono numeri reali, e i, j, k sono tre *unità immaginarie*. Hamilton definisce la "parte scalare" e la "parte vettoriale" di un quaternione:

$$\mathrm{S}\,Q \stackrel{\text{def}}{=} a \qquad \mathrm{V}\,Q \stackrel{\text{def}}{=} b\,i + c\,j + d\,k.$$

Se $\mathrm{V}\,Q = 0$, Q è detto "scalare." Se $\mathrm{S}\,Q = 0$, è detto "vettoriale." Si noti che la parte scalare di un quaternione può essere identificata con un reale, mentre la parte vettoriale può essere interpretata come un vettore (b, c, d) di $\mathbb{R}^3$.

Come nel caso dei complessi, tra quaternioni sono definiti *somma* e *prodotto*. La somma è ovvia: se

$$Q_1 = a_1 + b_1\,i + c_1\,j + d_1\,k \qquad Q_2 = a_2 + b_2\,i + c_2\,j + d_2\,k$$

abbiamo

$$Q_1 + Q_2 \stackrel{\text{def}}{=} (a_1 + a_2) + (b_1 + b_2)\,j + (c_1 + c_2)\,j + (d_1 + d_2)\,k.$$

Per il prodotto si parte definendo i prodotti delle unità immaginarie, come segue:

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ i\,j = -j\,i &= k \\ j\,k = -k\,j &= i \\ k\,i = -i\,k &= j. \end{aligned} \tag{A.1}$$

Dopo di che il prodotto tra due quaternioni generici si estende per linearità:

$$\begin{aligned} Q_1\,Q_2 = {} & (a_1\,a_2 - b_1\,b_2 - c_1\,c_2 - d_1\,d_2) + \\ & (a_1\,b_2 + a_2\,b_1 + c_1\,d_2 - c_2\,d_1)\,i + \\ & (a_1\,c_2 + a_2\,c_1 + d_1\,b_2 - d_2\,b_1)\,j + \\ & (a_1\,d_2 + a_2\,d_1 + b_1\,c_2 - b_2\,c_1)\,k. \end{aligned} \tag{A.2}$$

Si vede bene dalle (A.1), ma anche delle (A.2), che la moltiplicazione *non è commutativa.*

L'espressione (A.2) del prodotto, piuttosto complicata, si semplifica molto se Q_1, Q_2 sono vettoriali. In tal caso

$$\begin{aligned} Q_1\,Q_2 = {} & -(b_1\,b_2 + c_1\,c_2 + d_1\,d_2) + (c_1\,d_2 - c_2\,d_1)\,i + \\ & (d_1\,b_2 - d_2\,b_1)\,j + (b_1\,c_2 - b_2\,c_1)\,k. \end{aligned}$$

Se introduciamo (con notazione moderna) il vettore $\mathbf{v}_1$ di componenti (a_1, b_1, c_1) e analogamente $\mathbf{v}_2$, si vede subito che

$$\mathrm{S}\,(Q_1\,Q_2) = -\mathbf{v}_1 \cdot \mathbf{v}_2 \qquad \mathrm{V}\,(Q_1\,Q_2) = \mathbf{v}_1 \times \mathbf{v}_2.$$

Si definisce *coniugato* di un quaternione Q il $\bar{Q}$ tale che

$$\mathrm{S}\,\bar{Q} = \mathrm{S}\,Q \qquad \mathrm{V}\,\bar{Q} = -\mathrm{V}\,Q.$$

Osserviamo che

$$Q\,\bar{Q} = a^2 + b^2 + c^2 + d^2 \tag{A.3}$$

Per finire: a Hamilton è anche dovuta l'invenzione dell'operatore vettoriale ∇:

$$\nabla \stackrel{\mathrm{def}}{=} i\,\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} + k\,\frac{\partial}{\partial z}.$$

Applicando ∇ a un campo quaternionico puramente vettoriale, con la regola formale del prodotto, si ottiene

$$\mathrm{S}\,\nabla Q = -\nabla \cdot \mathbf{v} \qquad \mathrm{V}\,\nabla Q = \nabla \times \mathbf{v}.$$

Si ottengono quindi in un colpo solo divergenza e rotore del campo vettoriale $\mathbf{v}$ (attenzione però al segno "$-$" nella parte scalare).

Poniamoci un problema: esiste sempre l'inverso di un quaternione? Questo vuol dire, dato Q, se esiste Q' tale che $Q\,Q' = 1$ (inverso destro), oppure che $Q'\,Q = 1$ (inverso sinistro). Sebbene la moltiplicazione non sia commutativa, si dimostra facilmente che i due inversi, se esistono, debbono coincidere. Dalla (A.3) segue che se $Q \neq 0$ l'inverso esiste (ed è unico):

$$Q' = \frac{\bar{Q}}{a^2 + b^2 + c^2 + d^2}.$$

Pertanto i quaternioni formano un *campo non commutativo.*